σ-Frames.

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The interest in frames (= local lattices, complete Heyting algebras) has three main sources, not entirely unrelated to each other: first, they appear as the most natural lattice-theoretic generalization of topologies in which various topological notions and constructions can be expressed, besides being not merely a generalization but actually the class of all models of the laws which finitary meets and arbitrary joins in topologies satisfy; secondly, they provide the sites, in the sense of Grothendieck, for particularly natural types of sheaf categories, and, finally, they occur as the lattices of global sections of the subobject classifiers of arbitrary Grothendieck topoi. The less restrictive notion of σ-frame which only requires the existence of countable joins, may strike one at first sight as a merely formal variant of the concept of frame, but it actually has a distinctive significance of its own since σ-frames which fail to be frames naturally arise in various different contexts. Typical instances are given by the cozero set lattices of topological spaces and th Boolean σ-algebras which one encounters in topology, measure theory, and logic. It seems reasonable, then, to study σ-frames in their own right.

The first section of this paper deals with the compact regular σ-frames and establishes a remarkable similarity between these and compact regular frames, which were considered in Banaschewski–Mulvey [ ]. Thus, they are characterized ~~the compact regular σ-frames~~ as the lattices of cozero sets of compact Hausdorff spaces (Proposition 1) and then shown to be ~~the compact regular σ-frames are~~ coreflective in the category of all σ-frames (Proposition 2). The second section ~~adjointness between~~ is concerned with regular σ-frames. ~~and and frames is considered which is given by composing the coreflection to the regular σ-frames with the general adjointness between frames and σ-frames.~~ The main point here is that ~~the regular σ-frames~~ these are, on the one hand, exactly the largest regular σ-subframes of frames (Proposition 4) and, on the other, precisely the images of compact regular σ-frames (Proposition 6). The latter results from the interesting fact that the regular σ-frames turn out to be the same as the Alexandroff algebras of Reynolds [ ] (Proposition 5), and provides a new proof of the basic result ~~[illegible]~~ of [ ] which identifies the Alexandroff algebras as the cozero set lattices of arbitrary frames (Corollary 3, Proposition 6). ~~[illegible]~~ These facts about regular σ-frames should be ~~viewed in comparison~~ contrasted with the situation for frames. For the latter, regularity is a much weaker property: ~~[illegible] regular Hausdorff topology [illegible]~~ for instance, the topology of a regular Hausdorff space X is the image of a compact regular frame iff X is completely regular, ~~[illegible]~~ whereas for σ-frames, ~~have the property that~~ in a sense, regularity already implies complete regularity. A specific ~~[illegible]~~ feature of this is that the lattice of all regular ideals of a regular σ-frame L is the compact regular coreflection of the ~~[illegible]~~ frame freely generated by L (Proposition 7). One of the main purposes of the final section is to derive, in the present setting, the characterization of the compact regular Hausdorff spaces as those T_0-spaces which have an Alexandroff basis, due to Wasileski [ ] (Proposition 11). ~~As a step towards this, a special type of lattice The most important step towards this involves showing that for certain types of lattices, here called Alexandroff lattices~~ This naturally leads to the consideration of certain (finitary) lattices, called Alexandroff lattices here, which share with regular σ-frames the property that their regular ideal lattices are compact regular frames. The exact relationship between the latter and ~~[illegible]~~ Alexandroff lattices is described as a certain adjointness (Proposition 8), and the maximal regular filters of an Alexandroff lattice are shown to form a compact Hausdorff space whose topology is isomorphic to the regular ideal lattice (Proposition 9), i.e. they are identified as the frame spectrum of the latter. In addition, the bases of compact Hausdorff spaces which are closed under finite union and intersection are characterized as ~~[illegible]~~ the normal Alexandroff lattices (Proposition 10).

One point which has become of some interest in connection with frames (Johnstone [ ]), the rôle of the Axiom of Choice, ~~has~~ is not ~~[illegible]~~ systematically explored in the present context. It seems that σ-frames specifically invite the use of countably many dependent choices, and for several of our results we have no way of avoiding this. In particular although the coreflection to compact regular σ-frames itself does not require any choice, connecting this with frames seems to.

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The first part of this paper owes a great deal to the arguments contained in the original draft of Banaschewski-Mulvey [ ], written at Lewes, Sussex, in May 1978, and Christopher J. Mulvey's collaboration is gratefully acknowledged. The final version of [ ], as a result of extensive simplification, no longer contains the material needed here, and so it came as a pleasant surprise that ideas discarded as unnecessarily complicated in the case of frames turn out to be useful for σ-frames.

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Preliminaries

Recall that a *frame* (= complete Heyting algebra, local lattice) is a complete lattice satisfying the distribution law $x \wedge \bigvee x_\alpha = \bigvee x \wedge x_\alpha$ for any x and any family (x_α); in analogy, we call a σ-complete lattice in which $x \wedge \bigvee x_n = \bigvee x \wedge x_n$ for all x and all sequences x_n $(n = 1, 2, \dots)$ a *σ-frame*. For frames L and M, the maps $h : L \to M$ are maps of the underlying sets which preserve all finite meets and arbitrary joins; analogously, the requirement for maps of σ-frames is the preservation of all finite meets and countable joins. We let $\mathbf{Frm}$ and $\sigma\mathbf{Frm}$ be the corresponding categories.

In any lattice, a *filter* is a subset F such that $\bigwedge E \in F$ for any finite subset $E \subseteq F$ and $x \in F$ whenever $z \leq x$ for some $z \in F$. F is *prime* iff $\bigvee E \in F$ implies $E \cap F \neq \emptyset$ for all finite subsets E of the lattice, *σ-prime* iff this condition holds for all at most countable subsets E, and *completely prime* iff it holds for arbitrary E. Further, F is called *σ-open* iff $\bigvee x_n \in F$ implies $x_1 \vee \dots \vee x_k \in F$ for some k. Note that a filter F which is prime and σ-open is σ-prime. An *ideal* is a subset J satisfying the duals of the requirements for filters.

Of central importance in this paper is the following relation $\prec$, defined in any lattice with zero 0 and unit e such that $x \prec z$ (x is *rather below* z) iff $x \wedge y = 0$ and $z \vee y = e$ for some y. We note the following basic properties of $\prec$:

(1) $x \leq a \prec c \leq z$ implies $x \prec z$.

(2) if $a \prec c$ and $x \prec z$ then $a \wedge x \prec c \wedge z$ and $a \vee x \prec c \vee z$.

(3) Any zero and unit preserving lattice homomorphism preserves $\prec$.

A frame is called *regular* iff $x = \bigvee z\,(z \prec x)$ for each element x; similarly, we call a σ-frame regular iff each element x is the join of some sequence x_n $(n = 1, 2, \dots)$ such that $x_n \prec x$ for all n, and we express the latter by saying $x = \bigvee x_n\,(x_n \prec x)$. Also, a frame is called *compact* iff its unit e is compact in the usual lattice sense, i.e. $e = \bigvee D$ implies $e \in D$ for any up-directed set D; analogously, we call a σ-frame compact iff $e = \bigvee x_n$ implies $e = x_1 \vee \dots \vee x_k$ for some k.

These notions have an obvious topological significance: for topologies ΩX, $U \prec V$ means that V contains the closure $\overline{U}$ of U, and hence ΩX is regular iff X is a regular space. Similarly, ΩX is compact iff the space X is (quasi)compact.

The full subcategory **CRFrm** of **Frm** given by the compact regular frames is (1) dually equivalent to the category of compact Hausdorff spaces by the functor $\mathfrak{O}$ which assigns to each space X its lattice $\mathfrak{O}X$ of open sets, and (2) coreflective in **Frm** by the functor $\mathfrak{R}$ which assigns to each frame L the largest regular subframe of the ideal lattice of L, with the coreflection maps $\mathfrak{R}L \to L$ given by taking joins (Banaschewski - Mulvey [ ]). It should be noted that the proof of (1) requires the Axiom of Choice in some form whereas that of (2) does not (see also Johnstone [ ]).

The dual equivalence (1) is part of the general dual adjointness between **Frm** and the category of topological spaces in which the second functor Σ associates with each frame L its (frame) prime spectrum ΣL, the space whose points are the completely prime filters P in L and whose open sets are the sets $\Sigma_x = \{P \mid x \in P \in \Sigma L$ The analogous notion of prime spectrum for <u>σ-frames</u> is given by the space ΠL consisting now of the <u>σ-prime</u> filters in L, with the topology generated by the sets $\Pi_x = \{P \mid x \in P \in \Pi L\}$. Note that both, ΣL and ΠL, may also be viewed as spaces of maps in the respective categories: ΣL corresponds to the subspace of $(\Sigma 2)^{|L|}$ given by the frame maps $L \to 2$ where 2 is the two-element frame and L| the underlying set of L, and analogously for L. $\Sigma 2 = \Pi 2$, incidentally, s the Sierpinski space with three open sets.

The natural counterpart, for σ-frames, to the functor $\mathfrak{O}$ on the category of topological spaces is the functor $\mathfrak{C}$ for which $\mathfrak{C}X$ is the lattice of cozero sets of the space X, i.e. the sets $Coz(f) = \{x \mid f(x) \neq 0\}$ where f ranges over the continuous real-valued functions on X, and the map $\mathfrak{C}X \to \mathfrak{C}Y$ corresponding to a continuous map $Y \to X$ is given by taking inverse images. It is a familiar fact that $\mathfrak{C}X$ is a regular σ-frame for each space X, obviously compact whenever X is compact. We recall the details for the sake of completeness: $\mathfrak{C}X$ is closed under finite intersections since $Coz(f) \cap Coz(g) = Coz(fg)$ and the whole space is a cozero set; also, for any real-valued continuous functions f_n $(n = 1, 2, \ldots)$, if f is the continuous function defined by $f(x) = \sum 2^{-n} |f_n(x)| \wedge 1$ then $Coz(f) = \bigcup Coz(f_n)$ so that $\mathfrak{C}X$ is closed under countable unions. This makes $\mathfrak{C}X$ a σ-frame. Moreover, if $U = Coz(f)$ then $U = \bigcup U_n$ where $U_n = \{x \mid \frac{1}{n} < |f(x)|\}$; now $\{t \mid \frac{1}{n} < |t|\} \prec \{t \mid 0 < |t|\}$ in the topology $\mathfrak{O}\mathbb{R}$ of the real line $\mathbb{R}$, hence $U_n \prec U$ in $\mathfrak{O}X$ because the map $\mathfrak{O}\mathbb{R} \to \mathfrak{O}X$ corresponding to f preserves $\prec$, and therefore also $U_n \prec U$ in $\mathfrak{C}X$ since $\mathfrak{O}\mathbb{R} = \mathfrak{C}\mathbb{R}$. We note that the σ-frames $\mathfrak{C}X$ have

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a further important property, usually referred to as normality, but this does not have to be considered here because it happens to be a general property of regular σ-frames which will be established later (Proposition 5).

There are obvious maps, for any space X and any σ-frame L, associating with each $x \in X$ the σ-prime filter $\mathfrak{C}(x) = \{U \mid x \in U \in \mathfrak{C}X\}$ in $\mathfrak{C}X$, and with each $a \in L$ the basic open set Π_a of ΠL. The first of these is clearly a continuous map $X \to \Pi\mathfrak{C}X$ for any space X, but the second will, in general, not be a map $L \to \mathfrak{C}\Pi L$ ⌉ the Π_a are usually not cozero sets of ΠL. However, there is an important case when this happens, as will be seen later (Proposition 1).

⌉ since

We conclude this section with a comment concerning terminology. In various instances (e.g. Isbell [ ], Johnstone [ ]) authors have found it appropriate to formulate facts about the category **Frm** in terms of its dual, called the category of *locales*, which is intended to make the relation with the category of topological spaces, and with geometric morphisms between topoi, more suggestive. However, in the present context, in which the emphasis is very much on the algebraic features of objects which themselves are of a more algebraic nature, it seemed decidedly preferable to take the maps in their natural direction, ⌉ ~~This seems in keeping with general practice which is happy to accept, say, Stone Duality and Pontryagin Duality as dual equivalences rather than forcing them~~ We do not quite see the need for distinguishing the category **Frm** from its dual by renaming the objects in addition to reversing the order of the maps, and would be happy to use "locale" in place of "frame", but ~~then decided that~~ following [ ] in this regard might help to avoid confusion.

⌉ as was also done in Reynolds [ ].

next: A1

1. Compact regular σ-locales.

Our first aim is to characterize the lattices $\mathfrak{C}X$ of cozero sets of compact Hausdorff spaces X up to isomorphism, as the compact regular σ-locales, and since we have already noted that these $\mathfrak{C}X$ are of this type it remains to prove the converse. We do this by establishing the necessary facts in the following three lemmas.

Lemma 1. Every σ-open proper filter in a ~~compact regular~~ σ-locale L is an intersection of σ-prime filters.

Proof. Let F be any σ-open proper filter and $a \notin F$. Then, since unions of chains of σ-open filters are again σ-open filters, take a σ-open proper filter G maximal among all such filters which contain F and miss a. We claim G is σ-prime, and for this it is enough to prove that G is prime since it is already σ-open. Consider, then, any b and c such that $b \vee c \in G$ and assume $b \notin G$ and $c \notin G$. Now, by distributivity, $H = \{x \mid b \vee x \in G\}$ is a filter, evidently σ-open, properly larger than G since $G \subseteq H$ but also $c \in H$, and proper since $b \notin G$. Hence it follows that $a \in H$ and therefore $b \vee a \in G$. Now, repeating the same argument with a in place of b leads to the conclusion that $a = a \vee a \in G$, a contradiction. This proves G is prime, and it then follows that F is an intersection of σ-prime filters.

Corollary. The σ-prime filters of a compact regular σ-locale L separate the elements of L.

Proof. If $a \nleq b$ then, by regularity, there exists a $c \prec a$ for which $c \nleq b$. Now, $F = \{x \mid c \prec x\}$ is a filter, by the basic properties of $\prec$, such that $b \notin F$ and $a \in F$. Moreover, F is σ-open: if $C \subseteq L$ is a countable chain such that $\bigvee C \in F$ then $c \prec \bigvee C$, i.e. $c \wedge u = 0$ and $u \vee \bigvee C = e$ for some u, and by compactness one then has $u \vee x = e$ for some $x \in C$ which means that $c \prec x$ and hence $x \in F$. It now follows that there exists a σ-prime filter $P \supseteq F$ such that $b \notin P$, and since $a \in P$ this shows P separates a from b.

Lemma 2. For any compact regular σ-locale L, the space ΠL is compact Hausdorff.

Proof. Let P and Q be two distinct σ-prime filters in L so that, say, there exists an $a \in P$ not in Q. Then, by regularity, $a = \bigvee a_n$ $(a_n \prec a$ and since P is σ-prime, $a_n \in P$ for some n. Now, $a \wedge c = 0$ and $a_n \vee c = e$ for some c, and since $a_n \vee c \in Q$ but $a_n \notin Q$ we have $c \in Q$ by primeness. It then follows that Π_a and Π_c are disjoint neighbourhoods of P and Q, respectively, and hence ΠL is Hausdorff. To prove the compactness of ΠL, let $\mathcal{L}$ be any proper filter basis in ΠL and consider $F = \bigcup \bigcap \Lambda$ $(\Lambda \in \mathcal{L})$. ~~Since intersections and updirected unions of σ-open filters are σ-open filters, F is such a filter, and by Lemma 1 there exists a σ-prime filter $P \supseteq F$. Then for any $a \in P$, there exists a $c \prec a$ in P as was seen before, and one has $c \wedge b = 0$ and $a \vee b = e$ for some b. Therefore if Λ is a member of $\mathcal{L}$ such that $a \notin Q$ for each $Q \in \Lambda$ then $b \in Q$ for each $Q \in \Lambda$ by primeness, hence $b \in \bigcap \Lambda$ so that $b \in F$, and finally $b \in P$ — a contradiction since $c \in P$ and $c \wedge b = 0$~~. It follows that $\Lambda \cap \Pi_a \neq \emptyset$ for each $a \in P$, which says that P belongs to the closure of Λ, and since this holds for every $\Lambda \in \mathcal{L}$, P is a cluster point of the filter basis $\mathcal{L}$. Hence ΠL is compact.

⌉ proper because $\emptyset \notin \mathcal{L}$,

⌊ for some b.

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Lemma 3. For any compact Hausdorff space X, an open set U is a cozero set iff $U = \bigcup U_n$ $(U_n \prec U)$. ~~[illegible]~~

Proof. ⌈ given that U is of this type, there exists, for each n, a continuous real-valued function on X such that $0 \leq f_n(x) \leq 1$ for all $x \in X$, $f_n(x) = 1$ for $x \in U_n$ and $f_n(x) = 0$ for $x \notin U$, by Urysohn's Lemma and the fact that $V \prec U$ means $\overline{V} \subseteq U$ in any topology. Then it follows that the continuous function f, defined by $f(x) = \sum 2^{-n} f_n(x)$, has U as its cozero set. ~~Conversely, if U is the cozero set of a continuous real valued function f on X then $U = \bigcup U_n$ where $U_n = \{x \mid \frac{1}{n} < |f(x)|\}$, and since $\{x \mid \frac{1}{n} < |x|\} \prec \{x \mid 0 < |x|\}$ in $\mathfrak{O}R$ it follows that $U_n \prec U$ in $\mathfrak{O}X$.~~

⌈ That every cozero set has this property is an immediate consequence of the regularity of $\mathfrak{C}X$ which was proved earlier for arbitrary spaces. Conversely,

~~Now, consider the map $L \to \mathfrak{O}\Pi L$ given by $x \mapsto \Pi_x$. ⌉ this preserves finite meets and countable joins, and hence also the relation $\prec$. Therefore, which makes if $x = \bigvee x_n$ $(x_n \prec x)$ then $\Pi_x = \bigcup \Pi_{x_n}$ $(\Pi_{x_n} \prec \Pi_x)$ Π_x a cozero set of ΠL ⌊ Conversely, let $\Lambda \subseteq \Pi L$ be any cozero set. ⌈ so that, again by Lemma 3, $\Lambda = \bigcup \Lambda_n$ $(\Lambda_n \prec \Lambda)$, and if $\Lambda = \bigcup \Pi_x$ $(x \in S)$ for some subset S of L then each $\overline{\Lambda}_n \subseteq \Lambda$ is covered by finitely many Π_x, $x \in S$, by compactness, hence $\Lambda = \bigcup \Pi_x$ $(x \in T)$ for some countable $T \subseteq S$ and therefore $\Lambda = \Pi_a$ for $a = \bigvee T$. Thus the ⌉ map $x \mapsto \Pi_x$ actually maps L onto $\mathfrak{C}\Pi L$, and ⌋ by the Corollary of Lemma 1 this is an isomorphism.~~

~~Since the $P \in \Pi L$ are σ-prime filters,~~

~~⌊ by Lemma 3.~~

~~⌈ Then, as open set, $\Lambda = \bigcup \Pi_x$ $(x \in S)$ for some subset S of L, but also $\Lambda = \bigcup \Lambda_n$ $(\Lambda_n \prec \Lambda)$ by Lemma 3. Now, each $\overline{\Lambda}_n \subseteq \Lambda$ is cover by finitely~~

~~⌉ σ-locale~~

~~⌋ since it is one-one~~

A2'

Then F is a proper filter, and hence so is $G = \{x \mid z \succ x \text{ for some } z \in F\}$. Moreover, G is σ-open by the proof of the corollary of Lemma 1, and therefore there exists a σ-prime filter $P \supseteq G$. Now, for some $a \in P$ and $\Lambda \in \mathscr{L}$, suppose that $\Lambda \cap \Pi_a = \emptyset$. As was seen before, there exists a $b \succ a$ in P and then also a $c \succ b$; let $c \wedge x = 0$, $b \vee x = e$, $b \wedge y = 0$, and $a \vee y = e$ for suitable x and y. Then ~~[illegible]~~ for each $Q \in \Lambda$, $a \notin Q$ implies that $y \in Q$ so that $y \in \bigcap \Lambda$ and therefore $y \in F$; ~~or~~ on the other hand, $y \succ x$ so that $x \in G$ and hence $x \in P$, a ~~[illegible]~~ contradiction ~~that~~ since $c \in P$ and $c \wedge x = 0$. It follows ...

Now, consider the map $L \to \mathfrak{O}\Pi L$ given by $x \mapsto \Pi_x$. Since the $P \in \Pi L$ are σ-prime filters, this preserves finite meets and countable joins, and hence also the relation $\prec$. Therefore, if $x = \bigvee x_n (x_n \prec x)$ in L then $\Pi_x = \bigcup \Pi_{x_n} \ (\Pi_{x_n} \prec \Pi_x)$ in $\mathfrak{O}\Pi L$ which makes Π_x a cozero set of ΠL, for each $x \in L$. Conversely, let $\Lambda \subseteq \Pi L$ be any cozero set. Then $\Lambda = \bigcup \Pi_x (x \in S)$ for some subset $S \subseteq L$ since Λ is open, but also $\Lambda = \bigcup \Lambda_n (\Lambda_n \prec \Lambda)$ by Lemma 3. Now, each $\overline{\Lambda}_n \subseteq \Lambda$ is covered by finitely many Π_x, $x \in S$, by compactness, hence $\Lambda = \bigcup \Pi_x (x \in T)$ for some countable $T \subseteq S$, and therefore $\Lambda = \Pi_a$ where $a = \bigvee T$. This shows $x \mapsto \Pi_x$ maps L onto $\mathfrak{L}\Pi L$, and since the map is one-one by the corollary of Lemma 1 it is an isomorphism.

Hence we have proved:

Proposition 1. The compact regular σ-locales are, up to isomorphism, exactly the cozero set lattices of compact Hausdorff spaces.

This result can be slightly amplified as follows. The isomorphisms $L \to \mathfrak{L}\Pi L$ are natural in L and form one of the adjunctions for the adjoint pair of contravariant functors between the category of compact regular σ-locales and the category of compact Hausdorff spaces, given by $L \mapsto \Pi L$ and $X \mapsto \mathfrak{L}X$, the other adjunction being $X \to \Pi\mathfrak{L}X$ which maps each $x \in X$ to the σ-prime filter $\mathfrak{L}(x) = \{U \mid x \in U \in \mathfrak{L}X\}$. Clearly, the latter is a homeomorphism for each compact Hausdorff space X since $x \neq y$ implies $\mathfrak{L}(x) \neq \mathfrak{L}(y)$ and every σ-prime filter in $\mathfrak{L}X$ is $\mathfrak{L}(x)$ for its limit x. This says:

Corollary. The category of ~~complete~~ compact regular σ-locales is dually equivalent to the category of compact Hausdorff spaces, by the adjoint pair of contravariant functors $L \mapsto \Pi L$ and $X \mapsto \mathfrak{L}X$.

Remark. Proposition 1 and its corollary can be extended to cover the category of locally compact σ-compact Hausdorff spaces, i.e. locally compact Hausdorff spaces which are a countable union of compact subspaces (also called: locally compact Hausdorff spaces countable at infinity). The notion required for this on the side of σ-locales is the *way below* relation $\ll$ familiar from continuous lattices (Hofmann-Lawson [ ]). In the present setting, we define $a \ll c$ to mean: for any sequence (x_n), if $c \leq \vee x_n$ then there exists a k such that $a \leq x_1 \vee \dots \vee x_k$, and we call a σ-locale *continuous* iff $x = \vee x_n$ $(x_n \ll x)$ for each element x. A continuous regular σ-locale is then easily seen to be characterized by the condition that each of its elements is the join of a sequence of elements simultaneously rather below and way below it, and it is clear that compact regular σ-locales are of this type since $a \prec c$ implies $a \ll c$ by compactness. Now, we can prove the following: *The cozero set functor $\mathfrak{L}$ induces a dual equivalence between locally compact σ-compact Hausdorff spaces and continuous regular σ-locales.*

next : B1

Our next aim is to show that the compact regular σ-locales are coreflective in all σ-locales. We first consider the analogous fact for the regular σ-locales.

<u>Lemma 4.</u> <u>Any σ-locale L contains a largest regular σ-sublocale RL, and the correspondence $L \rightsquigarrow RL$ is functorial.</u>

Proof. The σ-sublocale M generated by any regular σ-sublocales $R_\alpha \subseteq L$ consists of all countable joins of elements $x_1 \wedge \ldots \wedge x_k$ where $x_i \in R_{\alpha_i}$ for some α_i. Then, for each i, $x_i = \bigvee x_{in}$ where $x_{in} \wedge z_{in} = 0$ and $x_i \vee z_{in} = e$ for some $x_{in}, z_{in} \in R_{\alpha_i}$, $n = 1, 2, \ldots$, and hence $x_1 \wedge \ldots \wedge x_k = \bigvee x_{1n_1} \wedge \ldots \wedge x_{kn_k}$ where

$$(x_{1n_1} \wedge \ldots \wedge x_{kn_k}) \wedge (z_{1n_1} \vee \ldots \vee z_{kn_k}) = 0, \quad (x_1 \wedge \ldots \wedge x_k) \vee (z_{1n_1} \vee \ldots \vee z_{kn_k}) = e$$

so that $x_{1n_1} \wedge \ldots \wedge x_{kn_k} \prec x_1 \wedge \ldots \wedge x_k$ in M. ~~It follows that~~ Consequently, each $x \in M$ is a countable join of elements in M which are rather below x in M, i.e. M is indeed regular. The remainder of the lemma follows immediately from the obvious fact that the image of a regular σ-locale L with respect to a σ-locale map $h: L \to M$ is clearly regular since h preserves the relation $\prec$.

Lemma 4 says that the regular σ-locales are coreflective in the category $\underset{\sim}{\sigma\text{Loc}}$, with coreflection functor R such that the coreflection maps $RL \to L$ are embeddings. The description of $RL \subseteq L$ given by the lemma is not very explicit Γ exactly which elements of L belong to RL. In some special cases, however, a simple direct Γ of the elements of RL is possible, as in the following

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<u>Example.</u> <u>If L is the underlying σ-locale of a compact regular locale then RL consists exactly of those $x \in L$ such that $x = \bigvee x_n$ $(x_n \prec x)$.</u> Clearly, every element of RL is of this type, and hence it remains to show that all such elements of L form a regular σ-locale $M \subseteq L$. Obviously, M is closed under ~~finite meets and~~ countable joins and finite meets (the latter by the basic properties of $\prec$), and hence a σ-locale, with its operations induced from L. The key to regularity is the fact that in L the relation $\prec$ interpolates, i.e. if $x \prec z$ then also $x \prec y \prec z$ for some y, Γ if $x \wedge u = 0$ and $z \vee u = e$ then already $y \vee u = e$ for some $y \prec z$ because z is the join of the updirected set of all such elements. Now, let $x = \bigvee x_n$ $(x_n \prec x)$ in L. For each n, take ⌋ $x_{n1} = x_n \prec x_{n2} \prec x_{n3} \prec \ldots \prec y_n \prec z_n \prec x_{n+1}$ by successively applying the fact just noted, and put $\overline{x}_n = \bigvee x_{nk}$ so that $\overline{x}_n \in M$. Next, take u_n and v_n in L such that $y_n \wedge u_n = 0$, $z_n \vee u_n = e$, $z_n \wedge v_n = 0$, $x_{n+1} \vee v_n = e$; then $v_n \prec u_n$ and one can define $\overline{v}_n \in M$ as the join of a sequence $v_{n1} = v_n \prec v_{n2} \prec v_{n3} \prec \ldots \prec u_n$. It follows that $\overline{x}_n \wedge \overline{v}_n \leq y_n \wedge u_n = 0$ and $x \vee \overline{v}_n \geq x_{n+1} \vee v_n = e$ so that $\overline{x}_n \prec x$ in M (and not merely in L), and since $x = \bigvee \overline{x}_n$ this proves the regularity of M. — Note that the L considered here are, up

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to isomorphism, exactly the σ-locales given by the topologies $\mathfrak{O}X$ of compact Hausdorff spaces (Banaschewski-Mulvey [ ]), and that $RL \subseteq L$ then corresponds exactly to the lattice $\mathfrak{C}X \subseteq \mathfrak{O}X$ of cozero sets of X, by Lemma 3.

Another functor we need, besides R, is the ideal lattice functor $\mathfrak{J}$ which associates with each σ-locale L the lattice $\mathfrak{J}L$ of countably generated ideals $J \subseteq L$, (which is a σ-locale since meet and countable join in $\mathfrak{J}L$ are the usual intersection and join of ideals) and with each map $h: L \to M$ of σ-locales the map that takes each $J \in \mathfrak{J}L$ to the ideal in M generated by the image $h(J)$. The restriction to countably generated ideals is natural in the present context since it ensures the existence of a σ-locale map $\mathfrak{J}L \to L$ for each L, given by taking joins, resulting in a natural transformation from $\mathfrak{J}$ to the identity functor.

Composing the functors R and $\mathfrak{J}$, we then have a natural transformation τ from $R\mathfrak{J}$ to the identity functor, $\tau_L : R\mathfrak{J}L \to L$ by join, and $R\mathfrak{J}L$ is compact regular for each L. We want to show this is the desired coreflection. The following two lemmas provide the main steps in this direction.

Lemma 5. For any compact regular σ-locale L, $\tau_L : R\mathfrak{J}L \to L$ is an isomorphism.

~~Proof. We begin by deriving a more explicit description of $R\mathfrak{J}L$ for the particular L at hand. Let $\mathcal{R}L \subseteq \mathfrak{J}L$ consist of the regular countably generated ideals of L, i.e. those $J \in \mathfrak{J}L$ with the property that $x \in J$ implies there exists a $y \in J$ such that $x \prec y$. $\mathcal{R}L$ is closed under the operations of $\mathfrak{J}L$, as easy calculation shows,~~

Proof. We explicitly construct an inverse for τ_L. For any $x \in L$, let $\gamma_L(x)$ be the ideal of all $z \prec x$ in L. Since $x = \bigvee x_n$ $(x_n \leq x_{n+1} \prec x)$, $z \prec x$ implies $z \wedge u = 0$ and $u \vee \bigvee x_n = e$ for some u, and by compactness it then follows that $u \vee x_n = e$ for some n, so that $z \leq x_n$; hence the ideal $\gamma_L(x)$ is generated by the x_n and thus $\gamma_L(x) \in \mathfrak{J}L$. Now, $\gamma_L(x \wedge y) = \gamma_L(x) \cap \gamma_L(y)$ for any $x, y \in L$ by the basic properties of $\prec$. On the other hand, if $z \prec x \vee y$ where $x = \bigvee x_n$ $(x_n \leq x_{n+1} \prec x)$ and $y = \bigvee y_n$ $(y_n \leq y_{n+1} \prec y)$ then, for some t, $z \wedge t = 0$ and $t \vee \bigvee x_n \vee \bigvee y_n = e$, hence $t \vee x_n \vee y_m = e$ for suitable n and m, and thus $z \leq x_n \vee y_m$. This shows $\gamma_L(x \vee y) \subseteq \gamma_L(x) \vee \gamma_L(y)$, and therefore $\gamma_L(x \vee y) = \gamma_L(x) \vee \gamma_L(y)$. Finally, for any

$x_1 \vee \ldots \vee x_k$

countable join $x = \bigvee x_n$, $z \prec x$ implies $z \prec x_n \vee .. \vee x_n$ for some k by compactness, and therefore $\gamma_L(x) \subseteq \bigvee \gamma_L(x_n)$ by the previous result. In all this shows γ_L is a map of σ-locales, and since L is regular one actually has $\gamma_L : L \to R\mathcal{J}L$. Now, very obviously $\tau_L\gamma_L(x) = x$ for each $x \in L$ by regularity. Regarding the other composite, we note that, for any $J \in R\mathcal{J}L$ and $x \in L$, $x \in J$ iff $x \prec \bigvee J$: $x \prec \bigvee J$ implies, by compactness, that $x \leq y$ for some $y \in J$ and hence $x \in J$; conversely, if $J = \bigcup J_n$ ($J_n \subseteq J_{n+1} \prec J$) then $x \in J$ implies that $x \in J_m$ for some n, and if $I \in R\mathcal{J}L$ is such that $J_n \wedge I = 0$ and $J \vee I = L$ then there exist $a \in J$ and $b \in I$ for which $a \vee b = e$, and one has $x \wedge b = 0$ so that $x \prec a$ and therefore also $x \prec \bigvee J$. This shows $\gamma_L\tau_L(J) = J$ for each $J \in R\mathcal{J}L$.

Lemma 6. For any σ-locales L and M where M is compact regular, if $g, h : M \to R\mathcal{J}L$ are σ-locale maps such that $\tau_L g = \tau_L h$ then $g = h$.

Proof. For any $x \in M$, let $x = \bigvee x_n$ ($x_n \leq x_{n+1} \prec x$) so that $h(x) = \bigcup h(x_n)$ ($h(x_n) \subseteq h(x_{n+1}) \prec h(x)$). Then, by the last part of the preceding proof, for any n there exists an $a \in h(x)$ such that $z \leq a$ for all $z \in h(x_n)$, and therefore $\bigvee h(x_n) \in h(x)$. It follows that $z \in h(x)$ iff $z \leq \bigvee h(x_n)$ for some n, i.e. $h(x)$ is generated by the $\bigvee h(x_n)$. Now, $\tau_L g = \tau_L h$ implies that $\bigvee g(x_n) = \bigvee h(x_n)$ for all n, and hence $h(x) = g(x)$.

Consider, now, any map $h : M \to L$ of σ-locales where M is compact regular. Then $R\mathcal{J}h : R\mathcal{J}M \to R\mathcal{J}L$, hence $R\mathcal{J}h\tau_M^{-1} : M \to R\mathcal{J}L$ such that $h = \tau_L R\mathcal{J}h\tau_M^{-1}$, by Lemma , and Lemma shows that this factorization of h through τ_L is unique. In all we have proved:

Proposition 2. The compact regular σ-locales are coreflective in the category σLoc, with coreflection functor $R\mathcal{J}$ and coreflection maps $\tau_L : R\mathcal{J}L \to L$.

Remark. For any topological space X, if its topology $\mathfrak{O}X$ is viewed as a σ-locale and the corresponding coreflection to the compact regular σ-locales by Proposition 1, as the cozero set lattice $\mathfrak{L}\tilde{X}$ of a compact Hausdorff space $\tilde{X}$, then the coreflection map $\mathfrak{L}\tilde{X} \to \mathfrak{O}X$ determines a continuous map $u : X \to \tilde{X}$. Now, any continuous map $f : X \to Y$, Y compact Hausdorff, induces a σ-locale map $\mathfrak{L}Y \to \mathfrak{O}X$ which uniquely factors through $\mathfrak{L}\tilde{X} \to \mathfrak{O}X$, and therefore one has a unique continuous map $\tilde{f} : \tilde{X} \to Y$ such that $f = \tilde{f}u$. It follows that $u : X \to \tilde{X}$ is the Stone-Čech compactification of X. In other words: For $\mathfrak{O}X$, considered as a σ-locale, the compact regular coreflection is $\mathfrak{L}\beta X$. We shall see later that this is only a special case of a more general fact.

2. Regular σ-locales.

We now consider the relationships between the categories $\sigma\mathbf{Loc}$ and $\mathbf{Loc}$, with particular attention to regularity. There are two obvious functors between $\mathbf{Loc}$ and $\sigma\mathbf{Loc}$: $U: \mathbf{Loc} \to \sigma\mathbf{Loc}$ forgetting all but the countable joins, and $\mathfrak{H}: \sigma\mathbf{Loc} \to \mathbf{Loc}$ which associates with each σ-locale L the lattice $\mathfrak{H}L$ of all its σ-ideals, i.e. the ideals $J \subseteq L$ closed under countable joins, and with each map $f: L \to M$ of σ-locales the map $\mathfrak{H}f: \mathfrak{H}L \to \mathfrak{H}M$ taking each σ-ideal J of L to the σ-ideal of M generated by the image $f(J)$. That $\mathfrak{H}L$ is in fact a locale depends on the following description of joins in $\mathfrak{H}L$: $x \in \bigvee J_\alpha$ iff $x = \bigvee x_n$ where $x_n \in J_{\alpha_n}$ for suitable α_n; this implies that the elements of a meet $J \wedge \bigvee J_\alpha$ are of the form $z \wedge \bigvee x_n = \bigvee z \wedge x_n$ where $z \in J$ and $x_n \in J_{\alpha_n}$ for some α_n, and the latter clearly belongs to $\bigvee J \cap J_\alpha$.

<u>Proposition 3.</u> <u>$\mathfrak{H}$ is left adjoint to U, preserves compactness and regularity, and reflects compactness.</u>

Proof. The front adjunction $\eta_L: L \to U\mathfrak{H}L$ is given by mapping each $x \in L$ to its principal ideal $\downarrow x = \{z \mid z \leq x\}$, and the back adjunction $\varepsilon_M: \mathfrak{H}UM \to M$ by taking joins: naturality is obvious, $\varepsilon_{\mathfrak{H}L}\mathfrak{H}\eta_L$ maps each σ-ideal J of L first to the σ-ideal generated by all $\downarrow x_j (x \in J)$ and then to the join of the latter, i.e. the union of the $\downarrow x_j (x \in J)$, which is J; and $U\varepsilon_M \eta_{UM}$ takes each $x \in M$ first to $\downarrow x$ and then to the join of the latter, which is x.

Now, let L be a compact σ-locale, and consider any relation $L = \bigvee J_\alpha$ in $\mathfrak{H}L$. This implies $e = \bigvee x_n$ for some $x_n \in J_{\alpha_n}$, hence $e = x_{n_1} \vee \dots \vee x_{n_k}$ by the compactness of L, and thus L is already the join of finitely many of the J_α. Further, if L is regular then, for any $J \in \mathfrak{H}L$, $J = \bigvee \downarrow z \;(\downarrow z \prec J)$ since any $x \in J$ is a join $x = \bigvee x_n (x_n \prec x)$ and $x_n \prec x$ implies $\downarrow x_n \prec J$. Finally, if $\mathfrak{H}L$ is compact then L is evidently compact since the front adjunction is an embedding and $U\mathfrak{H}L$ is compact if $\mathfrak{H}L$ is.

By composing $\mathfrak{H}$ and U, respectively, with the natural embedding of the subcategory $R\sigma\mathbf{Loc} \subseteq \sigma\mathbf{Loc}$ of regular σ-locales and the coreflection functor R one obtains the following

<u>Corollary.</u> <u>The restriction of the functor $\mathfrak{H}$ to $R\sigma\mathbf{Loc}$ is left adjoint to the functor $S = RU: \mathbf{Loc} \to \sigma\mathbf{Loc}$.</u>

F (Reynolds [ ])

Remark. The functor $\mathfrak{H}$ may also be described topos-theoretically as associating with each σ-locale L the locale of subobjects of the terminal object $\mathbf{1}$ of the category of sheaves of sets on L with respect to the Grothendieck topology given on L by all countable joins: the subsheaves of $\mathbf{1}$ are exactly those sheaves S for which $Sx = 1$ if $x \in J$ and $Sx = \emptyset$ if $x \notin J$, J some σ-ideal of L.

More detailed information regarding the connection between **RσLOC** and **LOC** is given in

$\mathfrak{H}SM \to M$

Proposition 4. The front adjunction $L \to S\mathfrak{H}L$ is an isomorphism for all $L \in$ **RσLOC**, and the back adjunction $\mathfrak{H}SM \to M$ is an isomorphism for all compact regular locales M.

of the Example

Proof. The map $x \mapsto \downarrow x$ from L to $S\mathfrak{H}L$ is obviously one-one. To see it is onto, we use the same argument as in the last part of the proof of Lemma : for any $J \in S\mathfrak{H}L$, $J = \bigcup J_n$ ($J_n \subseteq J_{n+1} \prec J$), hence there is an $a_n \in J$ such that $J_n \subseteq \downarrow a_n$ for each n, and therefore $J = \downarrow a$ for $a = \bigvee a_n$. Next, if M is a compact regular locale then SM is exactly the σ-locale of the Example following Lemma , i.e. it consists of all $x \in M$ such that $x = \bigvee x_n$ ($x_n \prec x$) in M. Now, for any $x \in M$, $x = \bigvee z$ ($z \prec x$), and for each $z \prec x$ we may take $\bar{z}$ as the join of a sequence $z_1 = z \prec z_2 \prec z_3 \prec \dots \prec x$ so that $\bar{z} \leq x$, $\bar{z} \in SM$, and $x = \bigvee \bar{z}$. Hence $x = \bigvee J$ for the σ-ideal of all $y \leq x$ in SM, which say the map $\mathfrak{H}SM \to M$ is onto. To see that it is one-one, note that, for any $J \in \mathfrak{H}SM$, $x \in J$ iff $x = \bigvee x_n$ where $x_n \in SM$ and $x_n \prec \bigvee J$: $x \in J$ implies $x = \bigvee x_n$ ($x_n \prec x$) in SM and then obviously $x_n \prec \bigvee J$. Conversely, if $x_n \prec \bigvee J$ then $x_n \wedge z = 0$, $(\bigvee J) \vee z = e$ for some z, hence also $y \vee z = e$ for some $y \in J$ by compactness, and thus $x_n \leq y$ which implies $x_n \in J$; it now follows that $x = \bigvee x_n \in J$. As a result, $\bigvee J = \bigvee I$ implies $J = I$ for any $J, I \in \mathfrak{H}SM$, which was to be proved.

locales of

This proposition represents the regular σ-locales as the regular coreflections of the σ-locales derived from locales, and the compact regular locales as the locales of σ-ideals of compact regular σ-locales. It should be added that, for any compact regular σ-locale L, $\mathfrak{H}L$ is, up to isomorphism, the only compact regular locale M such that $L \cong SM$: this isomorphism implies that $\mathfrak{H}L \cong \mathfrak{H}SM$, and the latter is isomorphic to M by the proposition. Of course, if L is presented, by Proposition 1, as the cozero set lattice $\mathfrak{L}X$ of some compact Hausdorff space X then $\mathfrak{H}L \cong \mathfrak{O}X$.

Corollary. The functor $\mathcal{G}RJS$ is the coreflection of all locales to the compact regular locales.

Proof. If $h: M \to L$ is a map of locales where M is ~~completely~~ compact regular then $Sh: SM \to SL$ factors uniquely through $\tau_{SL}: RJSL \to SL$, say $Sh = \tau_{SL}\bar{h}$, hence one has the commuting diagram

$$\begin{tikzcd}
 & \mathcal{G}RJSL \arrow[d, "\mathcal{G}\tau_{SL}"] \\
\mathcal{G}SM \arrow[ur, "\mathcal{G}\bar{h}"] \arrow[r, "\mathcal{G}Sh"'] \arrow[d, "\varepsilon_M"'] & \mathcal{G}SL \arrow[d, "\varepsilon_L"] \\
M \arrow[r, "h"'] & L
\end{tikzcd}$$

so that $h = (\varepsilon_L \mathcal{G}\tau_{SL})(\mathcal{G}\bar{h}\,\varepsilon_M^{-1})$, and a simple computation involving the adjunction identities shows this factorization is unique.

Remark 1. The argument showing that ε_M is an isomorphism for compact regular locales M uses the Axiom of Choice; on the other hand, the existence of the coreflection from all locales to the compact regular ones can be proved without this (Banaschewski - Mulvey [ ], Johnstone [ ]). We do not know whether the above corollary can be established without the Axiom of Choice.

Remark 2. An immediate consequence of Proposition 4 is that the compact regular locales and the compact regular σ-locales form equivalent categories, S and $\mathcal{G}$ providing an L

L explicit adjoint
= equivalence. Since
he proof of Proposition 4
s entirely selfcontained,
ncluding the special de-
scription of SM which only
nvolves a direct com-
parison with the definition
f S, this equivalence
ould now be used to
btain Proposition 1 as
a consequence of the
esult that the
ompact regular locales
re exactly the topologies
f compact regular
Hausdorff spaces
(Banaschewski - Mulvey [ ]).
It seemed preferable, how-
ever, to present a self-
ontained proof of
roposition 1.

In the following we take a closer look at the properties of individual regular σ-locales. First, we have a general result ~~about~~ special joins in arbitrary σ-locales.

Lemma 7. If a and b are elements in a σ-locale L such that $a = \bigvee a_n\ (a_n \prec a)$ and $b = \bigvee b_n\ (b_n \prec a)$ then there exist u and v in L for which $a \vee u = a \vee b = v \vee b$ and $u \wedge v = 0$

Proof. We may assume that $a_n \leq a_{n+1}$ and $b_n \leq b_{n+1}$ for all n. Then, take u_n and v_n such that

$$a_n \wedge u_n = 0,\ a \vee u_n = e,\ b_n \wedge v_n = 0,\ b \vee v_n = e$$

and put $u = \bigvee u_n \wedge b_n$, $v = \bigvee a_n \wedge v_n$. Now

$$a \vee u = a \vee \bigvee u_n \wedge b_n = \bigvee a \vee (u_n \wedge b_n) = \bigvee a \vee b_n = a \vee b,$$

and likewise $v \vee b = a \vee b$; moreover

$$u \wedge v = \bigvee u_n \wedge b_n \wedge a_k \wedge v_k \quad (n,k = 1,2,\dots)$$

which is zero since $u_n \wedge a_k = 0$ if $n \geq k$ and $b_n \wedge v_k = 0$ if $k \geq n$.

Recall that an Alexandroff algebra is ⌉ a regular σ-locale

⌉ defined as

satisfying the following normality condition (Reynolds [ ]):

(N) If $a \vee b = e$ then there exist u and v such that $a \vee u = e = v \vee b$ and $u \wedge v = 0$.

Lemma now shows that this condition in fact holds in any regular σ-locale, and hence we have

Proposition 5. The regular σ-locales are exactly the Alexandroff algebras.

Remark. It seems remarkable that the regularity of a σ-locale already implies normality, a fact apparently overlooked previously. This is clearly a feature of countable joins: a regular locale need not satisfy (N), as is shown by the topology of a regular Hausdorff space which only has constant real-valued continuous functions. Note, however, how a suitable countability condition changes the latter situation drastically: a 2nd countable regular Hausdorff space is normal, indeed metrizable — a fact which may actually be construed as a consequence of Proposition 5 since the topology of such a space, viewed as a σ-locale, is regular.

The normality of regular σ-locales implies that the relation $\prec$ interpolates: if $x \prec z$ and therefore $x \wedge t = 0$ and $z \vee t = e$ for some t then there exist u and v such that $u \vee t = e = z \vee v$ and $u \wedge v = 0$ which shows that $x \prec u \prec z$. This will be used in the next proof which establishes a further crucial property of regular σ-locales.

Proposition 6. For any regular σ-locale L, the map $\tau_L : R\mathfrak{J}L \to L$ is onto, and hence the regular σ-locales are exactly the images of compact regular σ-locales.

Proof. First, a more convenient description of $R\mathfrak{J}L$ is needed. An ideal $J \subseteq L$ is called regular iff $x \in J$ implies that $x \prec z$ for some $z \in J$. Finite intersections and arbitrary joins of regular ideals are easily seen to be regular ideals again, by the basic properties of $\prec$. In particular, then, the countably generated regular ideals of L form a σ-locale $\mathcal{R}L \subseteq \mathfrak{J}L$. We claim that $\mathcal{R}L = R\mathfrak{J}L$ for any regular L.

As a consequence of the fact that $\prec$ interpolates in such L, any join $x = \bigvee x_n$ $(x_n \prec x)$ can always be replaced by a join $x = \bigvee z_n$ $(z_n \prec z_{n+1})$, put $z_1 = x_1$ and then take z_{n+1} such that $x_1 \vee \dots \vee x_n \vee z_n \prec z_{n+1} \prec x$ in L, for each n. Now, let J be any countably generated regular ideal so that J

⌈ Then, for each n, let J_n be the ideal generated by a sequence $a_{n1} = a_n \prec a_{n2} \prec a_{n3} \prec \dots \prec a_{n+1}$, hich implies that $_n \in \mathfrak{R}L$ and $J =)J_n$. Further,

| From this, define I_n as the ideal enerated by a equence $c_{n1} = c_n \prec c_{n2} \prec c_{n3} \prec \dots \prec b_n$; it ollows that $I_n \in \mathfrak{R}L$, $_{n-1} \cap I_n = 0$ since $_n \wedge b_n = 0$, and $_{n+2} \vee I_n = L$ since $_{n+2} \vee c_n = e$, and hus $J_{n-1} \prec J_{n+2}$ n $\mathfrak{R}L$.

is generated by a sequence $a_1 \prec a_2 \prec \dots$. ⌈ ~~and hence $J = \bigcup \downarrow a_n$ where $\downarrow a = \{x \mid x \prec a\}$ is a regular ideal for any $a \in L$. Then~~ take b_n and c_n, for each n, such that $a_n \wedge b_n = 0$, $a_{n+1} \vee b_n = e$, $a_{n+1} \wedge c_n = 0$, $a_{n+2} \vee c_n = e$, and hence $c_n \prec b_n$. | ~~and hence $\downarrow a_n \cap \downarrow b_n = 0$, $\downarrow a_{n+2} \vee \downarrow b_n = L$ so that $\downarrow a_n \prec \downarrow a_{n+3}$ in $\mathfrak{R}L$~~ | This shows $\mathfrak{R}L$ is regular, and therefore $\mathfrak{R}L \subseteq R\mathfrak{J}L$. Conversely, for any $J \in R\mathfrak{J}L$, $J = \bigcup J_n$ ($J_n \prec J_{n+1}$) and by a familiar argument there then exist elements $a_n \in J_{n+1}$ such that ($x \prec a_n$ for all $x \in J_n$); therefore, J is regular and countably generated, and hence $R\mathfrak{J}L \subseteq \mathfrak{R}L$. In all this proves that $\mathfrak{R}L = R\mathfrak{J}L$.

It is now easy to see that the map $R\mathfrak{J}L \to L$ is onto: if $x = \bigvee x_n$ ($x_n \prec x_{n+1}$) then the x_n generate a regular ideal J such that $\bigvee J = x$. The remainder of the proposition is clear since images of regular σ-locales are regular.

Proposition 6 leads to particular consequences for the regular σ-locales SL in relation to other constructs involving L. If $\mathfrak{K} : \mathbf{Loc} \to \mathbf{Loc}$ is the coreflection functor to the compact regular locales, with coreflection maps $\kappa_L : \mathfrak{K}L \to L$, one has a natural isomorphism $\mathfrak{K}L \to \mathfrak{J}\mathfrak{R}SL$ by the corollary of Proposition 4 which induces a further natural isomorphism $S\mathfrak{K}L \to \mathfrak{R}SL$ by Proposition 3, and by the present proposition this implies

__Corollary 1.__ <u>For any locale L, the map $S\kappa_L : S\mathfrak{K}L \to SL$ is onto.</u>

Now, $S\mathfrak{K}L$ is essentially the σ-locale of cozero sets of the canonical compact Hausdorff space whose topology is isomorphic to $\mathfrak{K}L$ (i.e. the locale spectrum of $\mathfrak{K}L$), and therefore each $a \in S\mathfrak{K}L$ is the image of the set $C = \mathbb{R} - \{0\}$ for some locale map $h : \mathfrak{O}\mathbb{R} \to \mathfrak{K}L$. Thus, for any $c \in SL$, if $a \in S\mathfrak{K}L$ is mapped to c then $c = f(C)$ for the composite $f = \kappa_L h : \mathfrak{O}\mathbb{R} \to L$. Since, conversely, any locale map $\mathfrak{O}\mathbb{R} \to L$ actually goes into SL because $S\mathfrak{O}\mathbb{R} = \mathfrak{O}\mathbb{R}$, one has

__Corollary 2.__ <u>For any locale L, $SL \subseteq L$ consists exactly of the cozero sets of L, i.e. the images of $\mathbb{R} - \{0\}$ under locale maps $\mathfrak{O}\mathbb{R} \to L$.</u>

Finally, by Proposition 4, this implies

__Corollary 3__ <u>Every regular σ-locale is isomorphic to the σ-locale of cozero sets of some locale.</u>

Remark 1. The last two Corollaries are results due to Reynolds [ ], except that in [ ] the right adjoint to the functor $\mathfrak{G}$ on the regular σ-locales (≅Alexandroff algebras) is defined by the cozero sets of L so that its lattice-theoretic significance, which is inherent in the definition of our S, is not made explicit. In the present context, the key to Corollary 2 is really the fact that every regular σ-locale is the image of a compact one (not explicitly noted in [ ]), and Corollary 3 just results from the simple step in the proof of Proposition 4 showing that $L \rightarrow S\mathfrak{G}L$ is onto. The technical details of [ ] are circumvented here by the implicit use of Urysohn's Lemma for compact Hausdorff spaces.

Remark 2. (see B9-A)

The proof of Proposition 6 identifies the compact regular coreflection CL of regular σ-locales L as the lattice $\mathfrak{R}L$ of all countably generated regular ideals of L. It is therefore natural to determine the significance of the lattice $\mathfrak{J}L \supseteq \mathfrak{R}L$ of all regular ideals of L. The result is as follows:

Proposition 7. For any regular σ-locale L, there are natural isomorphisms $\mathfrak{R}L \cong S\mathfrak{J}L$ and, correspondingly, $\mathfrak{J}L \cong \mathfrak{G}\mathfrak{R}L$; further, for any locale M, $\mathfrak{J}SM \cong \mathfrak{C}M$, natural in M.

Proof. As has already been noted, $\mathfrak{J}L$ is closed under finite meets and arbitrary joins of ideals, which makes $\mathfrak{J}L$ a locale since the lattice of all ideals is one; also, $\mathfrak{J}L$ is compact for the same reason $\mathfrak{R}L$ is. Moreover, if $J \in \mathfrak{J}L$ and $x \in J$ then take $x \prec y \prec z$ in J and corresponding elements u and v (as in the proof of Proposition 6) such that $v \prec u$, $u \wedge x = 0$, and $z \vee v = e$, and note that $\downarrow x = \{t \mid t \prec x\}$, which is a regular ideal by the basic properties of $\prec$ and the fact that $\prec$ interpolates, has zero meet with $\downarrow u$ whereas $J \vee \downarrow u = L$. This shows that $\downarrow x \prec J$ in $\mathfrak{J}L$, and by the regularity of J it now follows that $J = \bigvee I\ (I \prec J)$ in $\mathfrak{J}L$, i.e. $\mathfrak{J}L$ is a regular locale. Further, by the Example after Lemma , we see that $S\mathfrak{J}L$ consists of all $J \in \mathfrak{J}L$ such that $J = \bigvee J_n\ (J_n \prec J)$ in $\mathfrak{J}L$; by familiar arguments we conclude that these are exactly the countably generated J, i.e. $S\mathfrak{J}L = \mathfrak{R}L$, and then by Proposition 4, we also have $\mathfrak{G}\mathfrak{R}L \cong \mathfrak{J}L$. Finally, if M is any locale then $\mathfrak{J}SM \cong \mathfrak{G}\mathfrak{R}SM \cong \mathfrak{G}CSM \cong \mathfrak{C}M$, the last step by the corollary of Proposition 4 and the preceding one by the proof of Proposition 6. Of course, all isomorphisms occurring in this entire discussion are natural.

continued B9-A

<u>Remark2</u> As mentioned earlier, the metrizability of 2nd countable regular Hausdorff spaces fits into the present context; it may be of interest to give an explicit argument for it in terms of σ-locales. Viewed as such, the topology $\mathfrak{O}X$ of any space of this kind is indeed regular since every $U \in \mathfrak{O}X$ is the union of the closures of suitable members of the given countable basis $\mathcal{B}$. Hence, by Proposition 6, the map $C\mathfrak{O}X \to \mathfrak{O}X$ is onto, and since $C\mathfrak{O}X$ is essentially the lattice of cozero sets of a compact Hausdorff space by Proposition 1 it follows that the σ-locale maps $\mathfrak{O}\mathbb{I} \to \mathfrak{O}X$, $\mathbb{I}$ the unit interval, are jointly onto. Then, the basis $\mathcal{B}$ is already covered by countably many such maps, and this implies there is an onto map $\mathfrak{O}(\mathbb{I}^{\omega}) \to \mathfrak{O}X$ of σ-locales which, in turn, determines an embedding $X \to \mathbb{I}^{\omega}$.

back to B9

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continued from B9

3. Alexandroff Lattices

We note from the proof of Proposition 6 that the regularity of the locale $\mathcal{T}L$ of all regular ideals of a regular σ-locale only depends on the fact that the relation $\prec$ interpolates and not on the presence of infinitary joins. This may be considered a motivation for the following definition: A lattice A will be called an <u>Alexandroff lattice</u> iff

(AL1) A is distributive and has a zero and a unit.

(AL2) The relation $\prec$ interpolates in A.

(AL3) Each $x \in A$ is the join of all $z \prec x$, i.e. for any $x, y \in A$, if $z \leq y$ for all $z \prec x$ then $x \leq y$.

cont'd p. B10

Clearly, regular σ-locales are Alexandroff lattices, and the latter ~~may~~ might be viewed as a finitary version of the former. Examples of Alexandroff lattices which need not be regular σ-locales are the topologies of normal Hausdorff spaces (and arbitrary Boolean algebras, where ~~[illegible]~~ $x \prec y$ iff $x \leq y$). We note that some of the following considerations remain valid if one only assumes (AL1) and (AL2); the force of (AL3) is to have enough occurrences of ~~[illegible]~~ $z \prec x$ in A.

We let ALatt be the category of Alexandroff lattices and those maps between them which preserve finite meets (i.e. the unit and binary meets) and zero, and make the following attempt at preserving $\vee$: if $x \prec z$ and $y \prec t$ then $h(x \vee y) \leq h(z) \vee h(t)$. Note that these maps preserve $\prec$: for any $a \prec b$, let $a \prec s \prec t \prec b$ and $v \prec u$ such that $a \wedge u = 0$ and $v \vee t = e$; then $h(a) \wedge h(u) = h(a \wedge u) = 0$ and $h(b) \vee h(u) \geqslant h(t \vee v) = e$ so that indeed $h(a) \prec h(b)$. Moreover, these maps do in fact form a category: if gh is a composite of such maps, the non-trivial part to check is the condition on joins; ~~[illegible]~~ thus, if $x \prec z$ and $y \prec t$, take $x \prec u \prec z$ and $y \prec v \prec t$ so that $h(x \vee y) \leq h(u) \vee h(v)$, ~~[illegible]~~ $h(u) \prec h(z)$, and $h(v) \prec h(t)$, and therefore

$$gh(x \vee y) \leqslant g(h(u) \vee h(v)) \leqslant gh(z) \vee gh(t),$$

which is the desired conclusion. With this we can ~~[illegible]~~ prove the following,

NEEDS MODIFICATION!
add
$x \prec h(a) \Rightarrow x \leq h(c)$
for some $c \prec a$.

Proposition 8. The correspondence $A \rightsquigarrow \mathfrak{J}A$ defines a functor $\mathfrak{J}$ from ALatt to the category CRLoc of compact regular locales which is left adjoint to the functor W: CRLoc $\to$ ALatt forgetting all infinitary joins, where $\mathfrak{J}A$ is now the compact regular locale ~~[illegible]~~ ~~[illegible]~~ ~~[illegible]~~ of all regular ideals of A, for any Alexandroff lattice A:

Proof. To see that $A \rightsquigarrow \mathfrak{J}A$ is functorial, we first note that, for any map $h: A \to B$ of Alexandroff lattices, the ideal of B generated by the image $h(J)$ of any regular ideal J of A is a regular ideal since h preserves $\prec$. Moreover, the resulting map $\mathfrak{J}A \to \mathfrak{J}B$ preserves finite meets because h does and $J \wedge I = \{x \wedge y \mid x \in J, y \in I\}$. Also, it clearly preserves updirected joins since these are just given by union. Finally, $J \vee I$ is mapped to the ideal generated by all $h(x \vee y)$, $x \in J$ and $y \in I$, but since $x \prec z$ and $y \prec t$ implies $h(x \vee y) \leq h(z) \vee h(t)$, and there are such $z \in J$ and $t \in I$ by regularity, this ideal is contained in the join of the ideals generated by $h(J)$ and $h(I)$ — which is exactly what is needed to see that $\vee$ is preserved.

It remains to prove the adjointness. We claim that the front adjunction $\eta_A : A \to W\mathfrak{J}A$ is given by $a \rightsquigarrow \downarrow a = \{x \mid x \prec a\}$, and the back adjunction ~~[illegible]~~ $\varepsilon_L : \mathfrak{J}WL \to L$ by taking joins. To begin with, $a \rightsquigarrow \downarrow a$ is indeed a map of the required kind: it ~~[illegible]~~

obviously preserves zero and unit, and $\downarrow(x \wedge y) = \downarrow x \cap \downarrow y$ by the basic properties of $\prec$; moreover, if $x \prec z$ and $y \prec t$ then $\downarrow(x \vee y) \subseteq \downarrow z \vee \downarrow t$ since $x \vee y \in \downarrow z \vee \downarrow t$ trivially. On the other hand, taking joins is evidently a locale map $\gamma WL \to L$, and the naturality of both η_A and ε_L is obvious. Now, $\varepsilon_{\gamma A} \gamma \eta_A$ maps $J \in \gamma A$ first to the ideal generated by all $\downarrow x$, $x \in J$, and this in turn to its union, which is J by the regularity of J. Similarly, $W\varepsilon_L \eta_{WL}$ maps $x \in L$ first to the regular ideal $\downarrow x$ and this to $\bigvee \downarrow x = x$, the latter by the regularity of L.

Remark 1. The front adjunction $\eta_A : A \to W\gamma A$ is an embedding since $\downarrow a = \downarrow b$ implies $a = b$ by (AL3). On the other hand, $\varepsilon_L : \gamma WL \to L$ is always an isomorphism, as was observed in Banaschewski-Mulvey [ ]. The proof of this consists in showing that, for any regular ideal J in L, $J = \downarrow \bigvee J$; here, $J \subseteq \downarrow \bigvee J$ results immediately from the regularity of L, and the other inclusion is obtained by compactness, as in the proof of Lemma . It follows from this that W is a full embedding of CRLoc into ALatt as a reflective subcategory. A somewhat peculiar aspect of this is that a map between compact regular locales which is a map of Alexandroff lattices must already be a locale map.

Remark 2. If an lexandroff lattice A appens to be a ocale then γA is ist the compact gular coreflection $\mathcal{R}A$ A: the latter is the rgest regular sublocale the ideal lattice of (Banaschewski-ulvey [ ]) so that $A \subseteq \mathcal{R}A$ since γA s regular. On the ler hand, the regularity $\mathcal{R}A$ shows, by an gument similar to e one used earlier n σ-locales, that ny $J \in \mathcal{R}A$ is a gular ideal so that $\mathcal{R}A \subseteq \gamma A$.

If we compose γ with the spectrum functor Σ of locales we obtain a contravariant functor $\Sigma\gamma$ from Alexandroff lattices to compact Hausdorff spaces where $\Sigma\gamma A$ is the space of completely prime filters in γA. Actually, these spaces have an alternative description which is rather more familiar in particular instances, such as the case of Boolean algebras or normal T_1 topologies, besides being much less convoluted.

We need a couple of lemmas to get to the desired result.

Lemma 8. For any Alexandroff lattice A, A and γA have isomorphic lattices of regular filters.

~~Proof. For any regular filter F in A, let $\tilde{F} = \{J \mid J \in \gamma A, J \cap F \neq \emptyset\}$. This is clearly a filter in γA, and if $a \in J \cap F$ then also $x \prec a$ for some $x \in J \cap F$ since F is regular, and hence~~

Proof. The map $\eta_A : A \to W\gamma A$ induces (i) a map $\mathfrak{F} \rightsquigarrow \bar{\mathfrak{F}}$ from the filters in γA to those in A by taking inverse images, i.e. $\bar{\mathfrak{F}} = \{x \mid \downarrow x \in \mathfrak{F}\}$, and (ii) a map $F \rightsquigarrow \tilde{F}$ in the opposite direction by taking direct images and then the filter generated by the resulting filter basis

Moreover, if $\mathfrak{F}$ is regular than so is $\overline{\mathfrak{F}}$: if $\downarrow x \in \mathfrak{F}$ then take $J \prec \downarrow x$ in $\mathfrak{F}$ which implies, by a familiar argument, that there exists a $z \in \downarrow x$ such that $J \subseteq \downarrow z$, and therefore $\downarrow z \in \mathfrak{F}$ and $z \prec x$. Similarly, if F is regular so is $\widetilde{F}$ since $x \prec y$ implies $\downarrow x \prec \downarrow y$. Now, $\widetilde{F}$ may also be described as $\{J \mid J \in \mathfrak{J}A, J \cap F \neq \emptyset\}$ for if $\downarrow x \subseteq J$ for some $x \in F$ then there is some $z \in \downarrow x$ belonging to F and hence $z \in J \cap F$, and, conversely, if $x \in J \cap F$ trivially $\downarrow x \subseteq J$. With this one has $\overline{\widetilde{F}} = \{x \mid \downarrow x \in \widetilde{F}\} = \{x \mid \downarrow x \cap F \neq \emptyset\} = F$, since F is a regular filter. On the other hand, $\widetilde{\overline{\mathfrak{F}}} = \{J \mid J \cap \overline{\mathfrak{F}} \neq \emptyset\} = \{J \mid \downarrow x \in \mathfrak{F} \text{ for some } x \in J\}$ which clearly shows $\widetilde{\overline{\mathfrak{F}}} \subseteq \mathfrak{F}$; conversely, if $J \in \mathfrak{F}$ and, correspondingly, $I \prec J$ for some $I \in \mathfrak{F}$ then there exists an $x \in J$ such that $I \subseteq \downarrow x$ and hence $\downarrow x \in \mathfrak{F}$, so that $J \in \widetilde{\overline{\mathfrak{F}}}$. This proves that $\widetilde{\overline{\mathfrak{F}}} = \mathfrak{F}$, and since both correspondences, $F \mapsto \widetilde{F}$ and $\mathfrak{F} \mapsto \overline{\mathfrak{F}}$, preserve order, the assertion follows.

The points of the space $\Sigma \mathfrak{J}A$ are the ~~maximal~~ completely prime filters in $\mathfrak{J}A$, and in order to make use of the previous lemma we have to relate these to the regular filters. This is done by

Lemma 9. In any compact regular locale, the maximal regular filters are exactly the completely prime filters.

Proof. Let P be any completely prime filter. Then, for any $x \in P$, the regularity condition $x = \bigvee z \, (z \prec x)$ implies that $z \in P$ for some $z \prec x$, and hence P is regular. Moreover, if $F \supset P$ is any proper regular filter and $x \in F - P$ then there exists a $z \prec x$ in F and hence one has $z \wedge y = 0$ and $x \vee y = e$ for some y; now $x \notin P$ but $x \vee y = e$ is in P so that, by primeness, $y \in P$ and therefore $0 = z \wedge y \in F$, a contradiction. This shows P is a maximal regular filter. Conversely, let Q be any maximal regular filter. We first show Q is prime: If $a \vee c \in Q$ and $a \notin Q$, consider $F = \{x \mid a \vee x \in Q\}$ which is a filter by distributivity. Then, for any $x \in F$, there exists a $z \prec a \vee x$ in Q, i.e. $z \wedge u = 0$ and $a \vee x \vee u = e$, and by ~~compactness and~~ regularity and compactness one then also has a $y \prec x$ such that $a \vee y \vee u = e$ which shows that $z \leq a \vee y$, ~~and~~ hence $a \vee y \in Q$ and thus $y \in F$. Therefore, F is a regular filter, and since $F \supseteq Q$ this shows $F = Q$ which implies that $c \in Q$. To complete the proof, let $\bigvee D \in Q$ for any updirected set D; then $x \prec \bigvee D$ for some $x \in Q$ and by compactness there exists a $y \in D$ such that $x \leq y$ and hence $y \in Q$.

Let $\mathbb{P}A$ now be the space of maximal regular filters in A, with basic open sets $P_a = \{Q \mid a \in Q \in \mathbb{P}A\}$ and consider the one-one onto map $Q \rightsquigarrow \tilde{Q}$ from $\mathbb{P}A$ to $\Sigma\gamma A$ which is provided by the preceding two lemmas. Recall that the topology of the space $\Sigma\gamma A$ consists of the sets $\Sigma_J = \{\mathfrak{M} \mid J \in \mathfrak{M} \in \Sigma\gamma A\}$ for the regular ideals J of A. With this, one now has

$$\{Q \mid Q \in \mathbb{P}A, \tilde{Q} \in \Sigma_J\} = \{Q \mid Q \in \mathbb{P}A, J \in \tilde{Q}\} =$$

$$\{Q \mid Q \in \mathbb{P}A, J \cap Q \neq \emptyset\} = \bigcup P_x \ (x \in J)$$

which shows this map is a homeomorphism. This proves:

Proposition 9. The space $\mathbb{P}A$ of maximal regular filters of an Alexandroff lattice A is compact Hausdorff and its topology is isomorphic to the lattice γA of regular ideals of A. Moreover, the correspondence $A \rightsquigarrow \mathbb{P}A$ provides a contravariant functor from ALatt to the category of compact Hausdorff spaces which is naturally equivalent to the functor $\Sigma\gamma$.

⊛ That $\mathbb{P}A$ is a compact Hausdorff space can also be derived directly from its definition and the defining properties of Alexandroff lattices, although this is not entirely on the surface; however, it is not at all clear how the functoriality of $\mathbb{P}A$ can be obtained in a similarly direct way. In any event, it is of interest to see the connection between the frame γA which arises here by analogy with regular σ-frames and the space $\mathbb{P}A$ which is the more familiar construct.

Remark 1. If X is a normal Hausdorff space, so that its topology $\mathfrak{D}X$ is an Alexandroff lattice, then this proposition and Remark 2 following Proposition 7 show that the space of maximal regular filters of $\mathfrak{D}X$ is the Stone-Čech compactification βX of X — a familiar fact concerning normal Hausdorff spaces. Similarly, for any completely regular Hausdorff space X, the space of maximal regular filters of its cozero set lattice $\mathfrak{C}X$ is βX since $\mathfrak{C}X = S\mathfrak{D}X$ and ⌊

⌊ therefore $\gamma\mathfrak{C}X \cong \mathfrak{R}\mathfrak{D}X$, as seen earlier.

Remark 2. ⊛
Of particular note are the normal Alexandroff lattices which satisfy the stronger normality condition (N) in place of (AL2), such as Boolean algebras, regular σ-locales, and normal T_1 topologies. A particular consequence of (N), which] happens to be equivalent with (N), is the following type of interpolation condition regarding $\succ$: if $x \succ y \vee z$ then there exist a $t \prec z$ such that $x \succ y \vee t$. To see this, consider any u such that $x \wedge u = 0$ and $y \vee z \vee u = e$; then there exist elements t and w for which $y \vee t \vee u = e = w \vee z$ and $t \wedge w = 0$, and hence $t \prec z$ and $x \succ y \vee t$.

] actually

We use this fact to obtain the following

Lemma 10 In a normal Alexandroff lattice, the maximal regular filters are exactly the minimal prime filters.

Proof. First, we show that the maximal regular filters are prime. If F is such a filter and $a \vee b \in F$ but $a \notin F$ then the filter $G = \{x \mid a \vee x \in F\}$ is regular for if $z \succ a \vee x$ in F then there exists a $y \prec x$ such that $z \succ a \vee y$ by normality, and it follows that $y \in G$. Now, since G is proper and $G \supseteq F$, we have $G = F$ and therefore $b \in F$. To see that F

Also: min prime = reg prime.

is in fact minimal prime we only have to note that, by the first part of the proof of Lemma , no regular filter can properly contain a prime filter. For the same reason, a prime filter which is regular must be a maximal regular filter, and hence the converse will follow if we show that every minimal prime filter is indeed regular. To this end, let P be an arbitrary prime filter and consider $Q = \{x \mid z \gtrsim x \text{ for some } z \in P\}$. Then Q is a filter, by the basic properties of $\gtrsim$, and regular since $\gtrsim$ interpolates. Moreover, if $x \vee y \in Q$, i.e. $z \gtrsim x \vee y$ for some $z \in P$, and $x \notin Q$ then by normality there exist $s \gtrsim x$ and $t \gtrsim y$ such that $z \gtrsim s \vee t$, and since $x \notin Q$ we have $s \notin P$ so that $t \in P$ and therefore $y \in Q$. This shows Q is a prime filter, and since $Q \subseteq P$ it follows that $Q = P$ if P is minimal prime, which makes P regular.

Remark. It follows from Lemma that the space PA, for a normal Alexandroff lattice, is the minimal prime filter space of A. By taking the complements of the minimal prime filters one obtains the space of maximal ideals (with the Zariski topology - he basic open sets are iven by missing an lement); finally, changing om A to its dual, this becomes the ultrafilter pace, with the Zariski opology, of the dual. Now, or a topological space X, X is normal and s dual is the zero set attice ZX of X. The ltrafilter space of ZX, owever, is just the amiliar description of X, as in Gillman-erison [ ].

The primeness of the maximal regular filters in normal Alexandroff lattices leads to a topological representation of such lattices which will now be derived. In the following, a basic ring (of open sets) of a topological space is a basis for the open sets of the space which is closed under finite unions and intersections.

Proposition 10. The normal Alexandroff lattices are, up to isomorphism, exactly the basic rings of compact Hausdorff spaces.

Proof. For any normal Alexandroff lattice A, the map $x \rightsquigarrow P_x$ from A into the topology of the space $\mathbb{P}A$ which gives the standard basis for this topology preserves all finite joins and meets, the join part of this specifically because the maximal regular filters are prime. [~~Moreover, if $a \not\leq b$ for $a, b \in A$~~ Moreover, this map is one-one since it corresponds, by the natural equivalence between the functors $\mathbb{P}$ and $\Sigma\gamma$, to the map $x \rightsquigarrow \dagger x$, which is one-one because of (AL3), as noted earlier.

Conversely, if $\mathfrak{O}$ is a basic ring of a compact Hausdorff space X then $U \gtrsim V$ holds in $\mathfrak{O}$ iff $\overline{U} \subseteq V$, the "if" part specifically resulting from compactness and the fact that $\mathfrak{O}$ is closed under finite unions; this proves (AL3), and the condition (N) is obtained in the same way. Since (AL1) holds by definition, $\mathfrak{O}$ is therefore a normal Alexandroff lattice.

Clearly, a compact Hausdorff space X will usually have B15 many different basic rings, and in general there is no natural way of chosing basic rings in $\mathcal{D}X$ other than $\mathcal{D}X$ itself. However, there is one special case in which a natural choice of a basic ring different from $\mathcal{D}X$ actually ~~a normal Alexandroff lattice~~, does occur: for a Boolean space X, the open-closed subsets of X form the essentially unique normal Alexandroff lattice A

ley: what abt $\mathcal{D}X$!!

such that $\mathbb{P}A \cong X$
and A is complemented.

Our last proposition deals with ~~We conclude with some considerations concerning~~ Alexandroff lattices inside locales. Following Wasileski [ ], a basis A of a locale L (i.e. L is generated by A with respect to $\wedge$ and $\vee$) will be called an Alexandroff basis ⌉ iff

⌉ of L

(AB1) A is closed under finite meets and joins.

(AB2) $\prec$ interpolates in A.

(AB3) Each $x \in A$ is the join, in L, of all $z \prec x$ in A.

Then, one has the following result, essentially due to Wasileski [ ]:

Proposition 11. The following are equivalent for a locale L:

(1) L is isomorphic to a completely regular topology.

(2) L is spatial and SL is a basis of L.

(3) L is spatial and has an Alexandroff basis.

Proof. (1) $\Rightarrow$ (2). SL is the σ-locale of cozero sets of L, and for a completely regular topology the cozero sets indeed form a basis.

(2) $\Rightarrow$ (3). Obvious since SL is then an Alexandroff basis of L.

(3) $\Rightarrow$ (1). Let X be a space with an Alexandroff basis $\mathfrak{A}$ for its topology. Then, for each $x \in X$, the filter $\mathfrak{A}(x) = \{U \mid x \in U \in \mathfrak{A}\}$ in $\mathfrak{A}$ is regular by (AB3) and clearly prime, and therefore maximal regular by the proof of Lemma . Hence, $x \rightsquigarrow \mathfrak{A}(x)$ is a map $X \to \mathbb{P}\mathfrak{A}$, and since $\mathfrak{A}$ is a basis of X one easily sees this is an embedding. This shows X is completely regular Hausdorff.

Remark. Reynolds [ ] gives an analogous characterization of the complete regularity of topologies in which the existence of an Alexandroff basis is replaced by the stronger one that L have a regular σ-sublocale as a basis. Regarding the proofs in [ ] and in Wasileski [ ], one should note that, although the ⌉ characterizations of complete regularity are purely in terms of the lattices of open sets, the arguments for them employ the space of real numbers. Here, the real numbers do not appear at all if one interprets "completely regular Hausdorff" as "embeddable into a compact Hausdorff space".

⌉ resulting